Writing and solving differential equations

Step 1 : Decide what the two variables are

~~Step 2: Is there a linear rel^n between them – if yes, ~~ use algebra, if no, use ~~calculus~~ calculus ~~to find a formula & use it~~

Step 2 : Find the increase in one for a small increase in the other (doesn't matter which way round)

Step 4 Convert into a differential equation in the form $\frac{dy}{dx} = f(x)$

Step 5 Solve to get $y = F(x) + c$

Step 6 Use given info to substitute for any arbitrary constants ~~write an equation and solve for c~~

Then rewrite the ~~equation~~ formula for y

Step 7 Substitute the given info and solve for the required info.

Maybe limber up with a few E-type problems and discussion of WASE

DE sol^ns

Areas, Volumes

Energy = variable force × distance – springs, charges, escape velocity etc

Drop off in atm pressure

Ice on a lake

Distance covered with non "uniform accel"

Radioactive decay & other types of decay & exponential growth

Air resistance

Circular motion, SHM Elliptical orbits?

2 APPROACHES TO DE's

$$\Delta E = F \Delta s$$

$$\int \Delta E$$

$$\sum \Delta E = \sum F \Delta s =$$

$$\int dE = \int F\, ds$$

$$\Delta E = F \Delta s$$

$$\frac{\Delta E}{\Delta s} = F$$

$$\frac{dE}{ds} = F = mgR^2 r^{-2}$$

$$E = -mgR^2 r^{-1} + c$$

DE FUNCTIONS

$\frac{dy}{dx} = f(x)$ eg area, volume, distance, energy ĉ var force
F(x) generally similar fn to f(x) unless f(x) = reciprocal

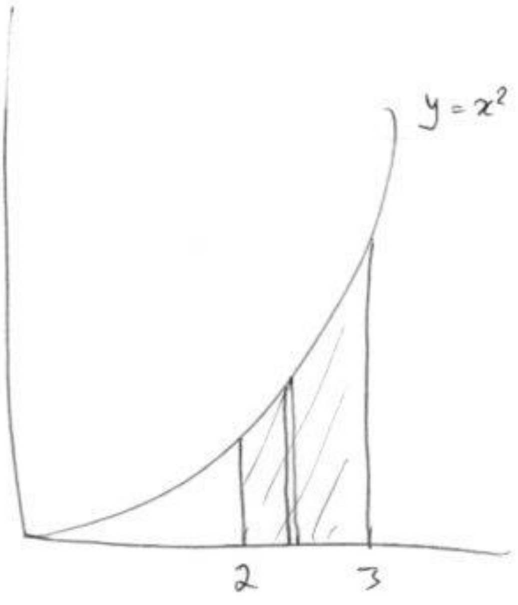
$\frac{dy}{dx} = f(y)$ if f(y) linear y = exp fn of x
eg exp growth/decay, atm pressure, air resistance

if f(y) is reciprocal y = √ fn of x

eg ice on lake

if f(y) is exp y = exp/log fn of x
if f(y) is trig y = trig fn of x
if f(y) is poly y = complicated rational fn of x

$$\boxed{\text{AREA}}$$



$$\Delta A = y\,\Delta x$$

$$\frac{dA}{dx} = y = x^2$$

$$A = \frac{x^3}{3} + c$$

If $x = 2$ $A = 0$

$$\therefore\ 0 = \frac{8}{3} + c$$

$$c = -\frac{8}{3}$$

$$\therefore\ A = \frac{x^3}{3} - \frac{8}{3}$$

when $x = 3$ $A = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$

$$\frac{dv}{dt} = g - kv$$

$$\int \frac{dv}{g-kv} = \int dt$$

$$\frac{dt}{dv} = \frac{1}{g-kv}$$

$$t = \frac{\log_e(g-kv)}{-k} + c$$

~~[illegible]~~

$\boxed{\text{Exponential Decay}}$

Substance decays at a rate proportional to amount present
200g at start 100g after 5 mins, How much after 8 mins

$$\frac{da}{dt} = -ka$$

$$\frac{dt}{da} = -\frac{1}{ka}$$

$$t = \frac{-1}{k} \log_e a + C$$

$$-kt = \log_e a + c$$

$$\cancel{a = e^{-kt}} + e^{c}$$

$$\cancel{e^{kt}} =$$

$$-c - kt = \log_e a$$

$$e^{-c-kt} = \cancel{\log} a$$

$$e^{-c} e^{-kt} = a$$

$$ce^{-kt} = a$$ 2 unknowns, 2 remaining bits of data

$$\boxed{a = \cancel{e}ce^{-kt}}$$ general ~~[illegible]~~ = "/ formula

~~[illegible]~~ $t = 0 \quad a = 200$

$$200 = \cancel{4}c \times 1$$

$$\therefore c = 200$$

$$\therefore a = 200e^{-kt}$$

$$t = 5 \quad a = 100$$

$$\therefore 100 = 200e^{-kt}$$

$\cancel{\tfrac{1}{2}} \cancel{2} e^{-}$

$1 = 2e^{-kt}$

$e^{kt} = 2$

$e^{5k} = 2$

$5k = \log_e 2$

$k = \dfrac{\log_e 2}{5}$

$= \cdot 139$

$\therefore \boxed{a = 200e^{-\cdot 139t}}$ - particular equation/formula

$t = 8 \Rightarrow a = 200e^{-1\cdot 1}$

$= 66g$

Problem: — To find pressure at 5000m

ATMOSPHERIC PRESSURE

Density at ground level = 1·29 kg/m^3

Pressure at ground level = 10^5 N m^{-2}

$g = 10$

Assume ~~..~~ T constant

At height h pressure = P

For a small ~~change~~ increase in height dh there is a small increase in pressure dp

$dp =$ ~~...~~ mass of dp m^3 of air × g

$= -g\,d\cancel{p}\,\frac{\rho}{10^5} \times 1{\cdot}29$

$dp = \frac{\cancel{..}\,1{\cdot}29\,g\rho}{10^5}\,dh$

$d\rho = -\frac{\rho}{\cancel{..}7752}\,dh$

$\frac{dh}{d\rho} = -\frac{\cancel{dp}\,\cancel{..}\,7752}{\rho}$

$h = -\cancel{..}^{7752}\log_e \rho + c$

$h = 0 \quad \rho = 10^5$

$\therefore\ 0 = -7752 \log_e \cancel{p}\,10^5 + c$

$\therefore\ c = 7752 \log_e 10^5 = 89247$

Alternative notation

$\int \frac{d\rho}{\rho} = -\frac{\cancel{p}1}{7752}\int dh$

$\log_e \rho = -\frac{h}{7752} + c$

$$\therefore \quad h = -7752 \log_e p + 89247$$

$$\frac{h - 89247}{-7752} = \log_e p$$

$$p = e^{\frac{h-89247}{7752}}$$

$$= 52458 \ \mathrm{Nm}^{-2}$$

To find height at which $p = 50\,000$

~~hect~~ $h = -7752 \log_e 50\,000 + 89247$

$$= 5372 \ \mathrm{m}$$

PROJECTILE WITH AIR RESISTANCE

~~8~~

A ball with a terminal velocity of 60 m/s is fired from a cannon at 100 m/s at an angle of elevation of 45°. How far away does it land?

Consider the vertical component

$$\frac{dv}{dt} = -g - \frac{v}{60}g$$

$$\cancel{\frac{dv}{dt} = -g\left(v + \frac{v}{60}\right)}$$

$$\frac{dt}{dv} = \frac{-1}{g + \frac{g}{60}v}$$

$$t = -\frac{60}{g}\log_e\left(g + \frac{g}{60}v\right) + c$$

$$-\frac{gt}{60} = \log_e\left(g + \frac{g}{60}v\right) + c$$

$$g + \frac{g}{60}v = Ce^{-gt/60}$$

$$\frac{g}{60}v = Ce^{-gt/60} - g$$

$$v = \frac{60C}{g}e^{-gt/60} - 60$$

$$v = Ce^{-gt/60} - 60$$

$$t = 0 \quad v = 100/\sqrt{2}$$

$$\therefore 100/\sqrt{2} = C - 60$$

$$C = 100/\sqrt{2} + 60 = 131$$

$$V = 131e^{-gt/60} - 60$$

~~s~~ first

$$\frac{ds}{dt} = 131e^{-gt/60} - 60$$

$$S = 131e^{-gt/60} \times \left(\frac{-60}{g}\right) - 60t + C$$

$$S = -933e^{-gt/60} - 60t + C$$

$$\cancel{S = -933e^{-gt/60} - 60t}$$

$$t = 0 \quad s = 0$$

$$0 = -933 + C$$

$$C = 933$$

$$\therefore S = 933 - 933e^{-gt/60} - 60t$$

$$S = 0$$

$$933 = 933e^{-gt/60} + 60t$$

$$933 = 933e^{-t/6} + 60t$$

$$1 = e^{-t/6} + .064t$$

$\theta = 10 \Rightarrow \cancel{.993}\ .828$

$t = \cancel{20}15 \Rightarrow 1.04$

$t = 14 \Rightarrow .993$

$t = 14.2 \Rightarrow 1.002$

$t = 14.15 \Rightarrow 1.0001$

$t = 14.15$ gives time in air

Now consider horizontal component

~~θ, $v = 2\ldots$~~

~~$s = 933 - 933e^{-9t/60} - 60t$~~

~~$t = 14.15$~~

$$\frac{dv}{dt} = -\frac{v}{60}g$$

$$\frac{dt}{dv} = -\frac{60}{gv}$$

$$t = -\frac{60}{g}\log_e v + C$$

$$-\frac{gt}{60} = \log_e v + C$$

~~$v = Ce^{-9t/60}$~~ $\log_e v = C - 9t/60$

$$v = e^{C - 9t/60}$$

~~$\frac{ds}{dt}$~~ $v = Ce^{-9t/60}$

$t = 0 \quad v = 70.7$

$70.7 = C$

$$\therefore v = 70.7e^{-9t/60}$$

$$\frac{ds}{dt} = 70.7e^{-9t/60}$$

$$s = -70.7 \times \frac{60}{9}e^{-9t/60} + C$$

$$s = -433e^{-t/6} + C$$

$t = 0 \quad s = 0$

$$\therefore 0 = \cancel{\ldots} - 433 + C$$

$$\therefore \cancel{824}\ C = 433$$

$$s = 433 - 433e^{-t/6}$$

$t = 14.15$

$$s = 433\left(1 - e^{-t/6}\right)$$

$$= 433\left(1 - e^{-14.15/6}\right)$$

$$= 392\text{ m}$$

displacement ~~position~~ at time $t = \begin{pmatrix} 433 - 433e^{-t/6} \\ 933 - 933e^{-t/6} - 60t \end{pmatrix}$

$\boxed{\text{Ice on lake}}$

$$\frac{da}{dt} \propto \frac{1}{a}$$

$$\frac{da}{dt} = \frac{k}{a}$$

$$\frac{dt}{da} = \frac{a}{k}$$

$$t = \frac{a^2}{2k} + c$$

$$a^2 = 2kt - c$$

$$t = 0 \quad a = 0$$

$$\therefore\ 0 = 0 - c$$

$$\therefore\ c = 0$$

$$a^2 = 2kt$$

$$a = \sqrt{2kt}$$

$$t = 10 \quad a = 5$$

$$5 = \sqrt{20k}$$

$$25 = 20k$$

$$k = 1.25$$

$$a = \sqrt{2.5t} = 1.58\sqrt{t}$$

This is $\frac{da}{dt} = \frac{k}{a}$ - reciprocal fn of a

→ square root solution.

$\boxed{\text{Extension}}$

Ice forming inside an insulated tank

$$\frac{da}{dt} \propto \frac{1}{a+k}$$

Need 3 bits of info to solve

eg thickness at 0hrs, 2hrs, 4hrs

$\boxed{\text{Abstract DE}}$

$$\left(1+\frac{dy}{dx}\right)^2 = \sin x \cos^2 x$$

$$1+\frac{dy}{dx} = (\sin x)^{\frac{1}{2}} \cos x$$

$$\frac{dy}{dx} = (\sin x)^{\frac{1}{2}} \cos x - 1$$

$$y = \frac{2}{3}(\sin x)^{3/2} - x + C$$

OTHER DE PROBLEMS

(1) Interest rate on a deposit is a function of amount ~~deposited~~ in the account

$$\frac{da}{dt} = \frac{a\sqrt{a}}{10} \quad \text{say}$$

or various other functions – lots of potential for varied problems here

(2) population growth $\frac{da}{dt} = ka$ ± other models

(3) Viscous flow over a flat surface 

Shear

(4) Calc^n of π by match & line method

(5)